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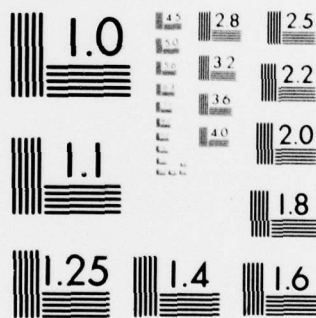UNCLASSIFIED            ORC-78-21                                            NL

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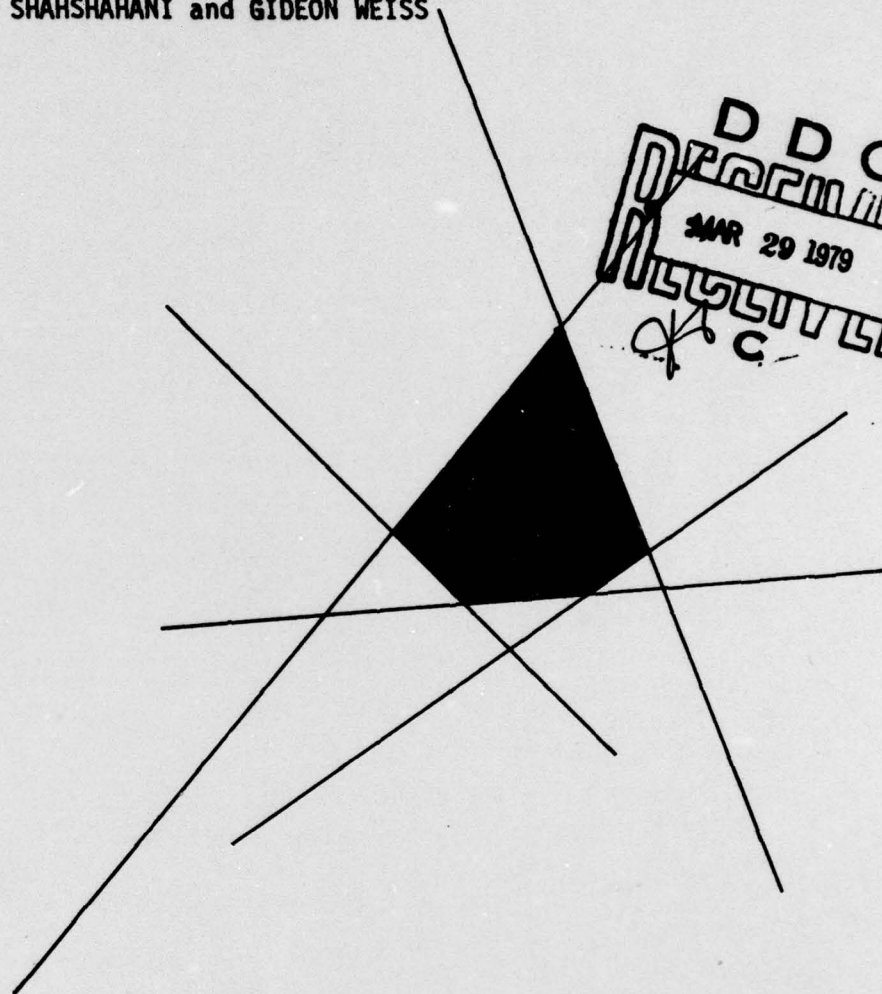# ON THE DURATION OF THE PROBLEM OF THE POINTS

by

SHELDON M. ROSS, MEHRDAD SHAHSHAHANI and GIDEON WEISS

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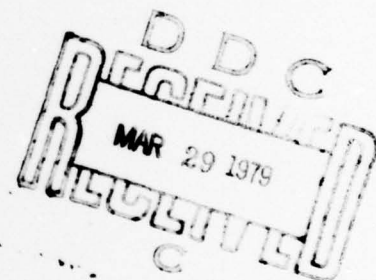ON THE DURATION OF THE PROBLEM OF THE POINTS

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(SEE ABSTRACT)

## ABSTRACT

We consider an r-player version of the famous problem of the points
which was the stimulus for the correspondence between Pascal and Fermat
in the seventeenth century.  At each play of a game, exactly one of
the players wins a point - player  i  winning with probability  $p_i$ .
The game ends the first time a player has accumulated his required
number of points - this requirement being  $n_i$  for player  i .  Our
main result is to show that  N , the total number of plays, is an
increasing failure rate random variable.  In addition, we prove some
Schur convexity results regarding  $P(N \leq k)$  as a function of  $\underline{p}$
(for  $n_i \equiv n$)  and as a function of  $\underline{n}$  (for  $p_i \equiv 1/r$) .

ON THE DURATION OF THE PROBLEM OF THE POINTS

by

Sheldon M. Ross, Mehrdad Shahshahani and Gideon Weiss

## 0.  INTRODUCTION AND SUMMARY

We are given an r-sided coin and numbers $n_1, n_2, \ldots, n_r$ along with
instructions to continue to flip the coin until side $i$ has appeared $n_i$
times for at least one $i$ . Each flip of the coin is assumed, independently
of other flips, to land on side $i$ with probability $p_i$ . Let the random
variable $N$ denote the number of flips that are performed. We are inter-
ested in studying the properties of $N$ .

In Section 1, we derive expressions for the mean and variance of $N$
and in Section 2 we show that $N$ has the increasing failure rate property--
namely that $P\{N = k + 1 \mid N > k\}$ is monotone nondecreasing in $k$ ,
$k = 0,1,2, \ldots$ . In Section 3, we show that $P\{N \leq k\}$ is a Schur convex
function of $(n_1, \ldots, n_r)$ when $p_i \equiv 1/r$ and is a Schur concave function
of $(p_1, \ldots, p_r)$ when $n_i \equiv n$ .

## 1. MEAN AND VARIANCE OF N

Assume that the flips are not performed at fixed times but rather at times chosen in accordance with a Poisson process with rate $\lambda = 1$. In addition, let us imagine that this process of coin-flipping continues indefinitely (even after some side has appeared the required number of times). Letting $T_i$ denote the time of the $n_i$th appearance of side $i$, then $T$, the length of time of the experiment, can be expressed as

$$(1) \qquad T = \min_{i=1,\ldots,r} T_i .$$

Now it follows from well-known facts about the Poisson process that the $T_i$ are independent gamma random variables with respective parameters $(n_i, p_i)$, $i = 1, \ldots, r$. Hence,

$$E[T] = \int_0^\infty P\{T > t\}dt$$

$$(2)$$

$$= \int_0^\infty \prod_{i=1}^r \int_t^\infty p_i e^{-p_i s} \frac{\left(p_i s\right)^{n_i - 1}}{(n_i - 1)!} \, ds .$$

Now the relationship between $T$ and $N$, the number of flips required, is that

$$(3) \qquad T = \sum_{i=1}^N X_i$$

where the $X_i$ are independent exponential random variables having rate 1 which are also independent of $N$. Thus, from (3), we have

$$E[T] = E[N]$$

which in conjunction with (2) yields an expression for $E[N]$ .

The variance of $N$ can also be obtained in a similar fashion. Namely from (3), upon conditioning, we obtain

$$Var \; [T] = E[N] \; Var \; [X] + E^2[X] \; Var \; [N]$$

implying that

$$Var \; [N] = Var \; [T] - E[N]$$

and $Var \; [T]$ can be obtained from the representation (2).

## 2.   N   IS AN INCREASING FAILURE RATE (IFR) RANDOM VARIABLE

### Theorem:

$N$ is an increasing failure rate random variable in the sense that

$P\{N = k + 1 \mid N > k\}$ is nondecreasing in $k$, $k = 0,1, \ldots$ .

### Proof:

The proof is by induction on $r$. For $r = 1$, $N = n_1$ is constant and hence is IFR. We now assume IFR for $r - 1$; in particular, we assume that $N'$, the number of required flips for an experiment with a coin having sides $2, \ldots, r$, integers $n_2, \ldots, n_r$, and probabilities $\dfrac{p_2}{1 - p_1}, \ldots, \dfrac{p_r}{1 - p_1}$, is IFR.

Letting $D_k$ denote the side obtained on the kth flip and $X_i(k)$ the *number of occurrences of side* $i$ in the first $k$ flips, we note that

$$P(N = k + 1 \mid N > k) = \sum_{i=1}^{r} P(D_{k+1} = i \, , \, X_i(k) = n_i - 1 \mid N > k)$$

$$= \sum_{i=1}^{r} p_i P(X_i(k) = n_i - 1 \mid N > k)$$

since $D_{k+1}$ is independent of $X_i(k)$ and of the event $N > k$.

Thus, it is enough to show that for all $i$, $P(X_i(k) = n_i - 1 \mid N > k)$ is nondecreasing in $k$. Obviously, it is enough to consider $i = 1$, and as $P(X_1(k) = n_1 - 1 \mid N > k) = 0$ for $k = 0,1, \ldots, n_1-2$, we need only consider $k \geq n_1 - 1$.

We use the definition of $N'$ to write

$$P(X_1(k) = n_1 - 1 \mid N > k) = \frac{P(X_1(k) = n_1 - 1 , N > k)}{\sum_{j=0}^{n_1-1} P(X_1(k) = j , N > k)}$$

$$= \frac{P(X_1(k) = n_1 - 1)P(N > k \mid X_1(k) = n_1 - 1)}{\sum_{j=0}^{n_1-1} P(X_1(k) = j)P(N > k \mid X_1(k) = j)}$$

$$= \frac{P(X_1(k) = n_1 - 1)P(N' > k - n_1 + 1)}{\sum_{j=0}^{n_1-1} P(X_1(k) = j)P(N' > k - j)} \; .$$

To show that this expression is nondecreasing in $k$ , it is enough to show that for $0 \le j \le n_1 - 1$ ,

$$\frac{P(X_1(k) = j)P(N' > k - j)}{P(X_1(k) = n_1 - 1)P(N' > k - n_1 + 1)}$$

is nonincreasing in $k$ (where $k \ge n_1 - 1$) .

The assumption that $N'$ is IFR implies $P(N' > k - j)/P(N' > k - n_1 + 1)$ is nonincreasing in $k$ . Finally, for $0 \le j \le n_1 - 1 \le k$ ,

$$\frac{P(X_1(k) = j)}{P(X_1(k) = n_1 - 1)} = \frac{\binom{k}{j}p_1^j\left(1 - p_1\right)^{k-j}}{\binom{k}{n_1 - 1}p_1^{n_1-1}\left(1 - p_1\right)^{k-n_1+1}}$$

$$= \left(\frac{1 - p_1}{p_1}\right)^{n_1-1-j} \frac{(n_1 - 1)!(k - n_1 + 1)!}{j!(k - j)!}$$

is immediately seen to be nonincreasing in $k$ .$\|$

## 3. SCHUR CONVEXITY OF $P\{N \leq k\}$

For an $r$ vector $\underline{x} = (x_1, \ldots, x_r)$, we denote by $x_{(i)}$ the $i^{th}$ largest component of $\underline{x}$. We say that the permutation invariant function $f$ is a Schur convex function if $f(\underline{x}) \geq f(\underline{y})$ whenever $\underline{x}$ majorizes $\underline{y}$ $\left(\text{written } \underline{x} \underset{m}{\geq} \underline{y}\right)$ where $\underline{x} \underset{m}{\geq} y$ if $\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}$, $j = 1, \ldots, r - 1$ and $\sum_{1}^{r} x_{(i)} = \sum_{1}^{r} y_{(i)}$. If the inequality between $f(\underline{x})$ and $f(\underline{y})$ is reversed, we say that $f$ is Schur concave.

### Proposition 1:

If $n_i \equiv n$, $i = 1, \ldots, r$, then $P\{N \leq k\}$ is a Schur convex function of $\underline{p} = (p_1, \ldots, p_r)$ for each $k$.

### Proof:

Consider first the case when $r$ equals 2. Then

$$P\{N \leq k\} = \begin{cases} 0 \ , \ 0 \leq k < n \\ \sum_{j=n}^{k} \binom{k}{j}[p^j(1 - p)^{k-j} + (1 - p)^j p^{k-j}] \ , \ n \leq k < 2n - 1 \\ 1 \ , \ 2n - 1 \leq k \ . \end{cases}$$

Differentiating with respect to $p$, when $n \leq k < 2n - 1$, we obtain

$$\frac{d}{dp} P\{N \leq k\} = k\binom{k - 1}{n - 1}[P(1 - p)]^{k-n}[p^{2n-1-k} - (1 - p)^{2n-1-k}]$$

which is positive for $p > 1/2$ thus implying that $P\{N \leq k\}$ is Schur convex when $r = 2$.

For the case of general $r$, it is well known (see [1], p. 47) that it suffices to show that

$$P\{N(p_1, \ldots, p_r) \leq k\} \geq P\{N(q_1, q_2, p_3, \ldots, p_r) \leq k\}$$

when $p_1 \geq p_2$, $q_1 \geq q_2$, $p_1 > q_1$, $p_1 + p_2 = q_1 + q_2$. Now, let us suppose that the two experiments (the first in which the outcomes occur with probabilities $(p_1, p_2, p_3, \ldots, p_r)$ and the other in which they occur with probabilities $(q_1, q_2, p_3, \ldots, p_r)$) are performed by first flipping a coin having $r - 1$ possible outcomes with probabilities $(p_1 + p_2, p_3, \ldots, p_r)$. If the outcome having probability $p_i$, $i = 3, \ldots, r$ occurs, then we say that outcome $i$ was the result for both experiments. If the outcome having probability $p_1 + p_2$ occurs, then for experiment 1 we determine its outcome (either 1 or 2) by flipping a coin having respective probabilities $\dfrac{p_1}{p_1 + p_2}$ and $\dfrac{p_2}{p_1 + p_2}$; whereas in experiment 2 we flip a coin whose probabilities are $\dfrac{q_1}{q_1 + q_2}$ and $\dfrac{q_2}{q_1 + q_2}$. Now, by conditioning on the number of the first $k$ flips that the coin (with $r - 1$ possible outcomes) results in outcome 1, we reduce the problem to the case $r = 2$, and so the proof is complete. $\|$

Let us assume that the $p_i$ are constant, then there is also a Schur result when the $n_i$ are allowed to be distinct.

## Proposition 2:

If $p_i = 1/r$, $i = 1, \ldots, r$, then $P\{N \leq k\}$ is a Schur convex function of $\underline{n} = (n_1, \ldots, n_r)$ for each $k$.

Proof:

Again consider first the case when $r = 2$. As

$$P\{N \le k\} = \left(\frac{1}{2}\right)^k \left[ \sum_{j=n_1}^{k} \binom{k}{j} + \sum_{j=n_2}^{k} \binom{k}{j} \right], \quad k \le n_1 + n_2 - 1$$

we must show that

$$\sum_{j=n_1}^{k} \binom{k}{j} + \sum_{j=n_2}^{k} \binom{k}{j} \ge \sum_{j=n_1-1}^{k} \binom{k}{j} + \sum_{j=n_2+1}^{k} \binom{k}{j}$$

where $k \le n_1 + n_2 - 1$, $n_1 > n_2$. The above inequality reduces to

$$\binom{k}{n_2} \ge \binom{k}{n_1 - 1}$$

which is easily verified to hold under the above conditions. The general case follows exactly as in Proposition 1.||

## 4. FINAL COMMENTS

The model considered has applications in reliability theory. Namely, consider an $r$ component system in which each component is subject to shocks. Every shock affects exactly one of the components--it affects component $i$ with probability $p_i$. Component $i$ can absorb at most $n_i - 1$ shocks before failing (one possibility being that with probability $p_i$ a shock knocks out the component in position $i$ for which there are a total of $n_i - 1$ spares). Assuming that the system structure is a series structure which means that the system is failed when at least one component is failed, it follows that $N$ represents the number of shocks required to cause system failure.

Of course, the model is an r-player version of the famous problem of the points which was the stimulus for the interchange of letters between Pascal and Fermat in the seventeenth century. They were mainly concerned with the probability of each player winning when $n = 2$ (winning means that $n_1$ type 1 events occur before $n_2$ type 2 events). In the r-player version, the probability of player $j$ winning can be expressed as

$$P\{j \text{ wins}\} = P\{T_j = \min(T_1, \ldots, T_r)\}$$

$$= \prod_{i=1}^{r} \frac{p_i^{n_i}}{(n_i - 1)!} \int_0^\infty \left[ \prod_{i \neq j} \int_t^\infty e^{-p_i s} s^{n_i - 1} \, ds \right] e^{-p_j t} t^{n_j - 1} \, dt \; .$$

# REFERENCE

[1]   Hardy, G. H., J. E. Littlewood and G. Polya, INEQUALITIES,
      Cambridge University Press, Cambridge, MA, 1952.